NSW GOVERNMENT

NSW Education Standards Authority

**2018** HIGHER SCHOOL CERTIFICATE EXAMINATION

# Mathematics Extension 1

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| **General Instructions** | • Reading time – 5 minutes<br>• Working time – 2 hours<br>• Write using black pen<br>• Calculators approved by NESA may be used<br>• A reference sheet is provided at the back of this paper<br>• In Questions 11–14, show relevant mathematical reasoning and/or calculations |
| **Total marks: 70** | **Section I – 10 marks** (pages 2–5)<br>• Attempt Questions 1–10<br>• Allow about 15 minutes for this section<br><br>**Section II – 60 marks** (pages 6–13)<br>• Attempt Questions 11–14<br>• Allow about 1 hour and 45 minutes for this section |

## Section I

**10 marks**
**Attempt Questions 1–10**
**Allow about 15 minutes for this section**

Use the multiple-choice answer sheet for Questions 1–10.

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**1** The polynomial $2x^3 + 6x^2 - 7x - 10$ has zeros $\alpha$, $\beta$ and $\gamma$.

What is the value of $\alpha\beta\gamma(\alpha + \beta + \gamma)$?

A. $-60$

B. $-15$

C. $15$

D. $60$

**2** The acute angle between the lines $y = 3x$ and $y = 5x$ is $\theta$.

What is the value of $\tan\theta$?

A. $\frac{1}{8}$

B. $\frac{1}{7}$

C. $\frac{1}{2}$

D. $\frac{4}{7}$

**3** What is the value of $\lim_{x \to 0} \frac{\sin 3x \cos 3x}{12x}$?

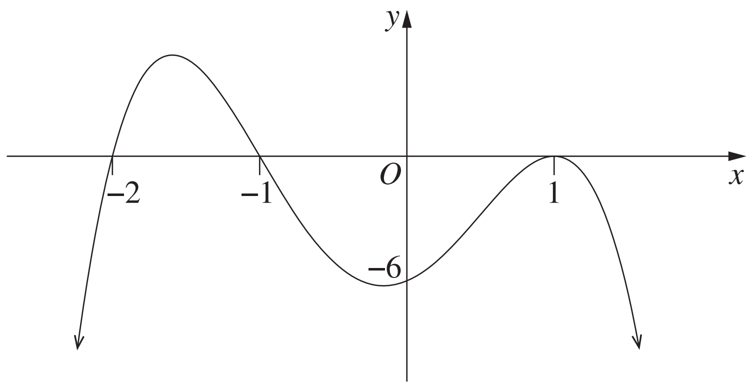
A. $\frac{1}{4}$

B. $\frac{1}{2}$

C. $\frac{3}{4}$

D. $1$

**4** The diagram shows the graph of $y = a(x+b)(x+c)(x+d)^2$.




What are possible values of $a$, $b$, $c$ and $d$?

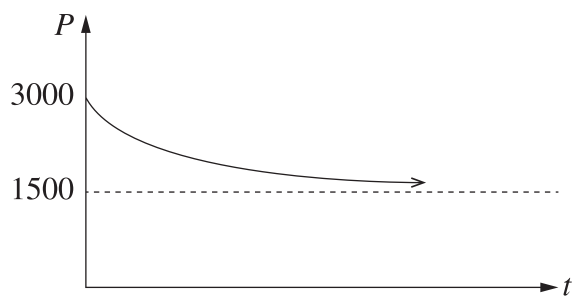
A. $a = -6,\ b = -2,\ c = -1,\ d = 1$

B. $a = -6,\ b = 2,\ c = 1,\ d = -1$

C. $a = -3,\ b = -2,\ c = -1,\ d = 1$

D. $a = -3,\ b = 2,\ c = 1,\ d = -1$

**5** The diagram shows the number of penguins, $P(t)$, on an island at time $t$.




Which equation best represents this graph?

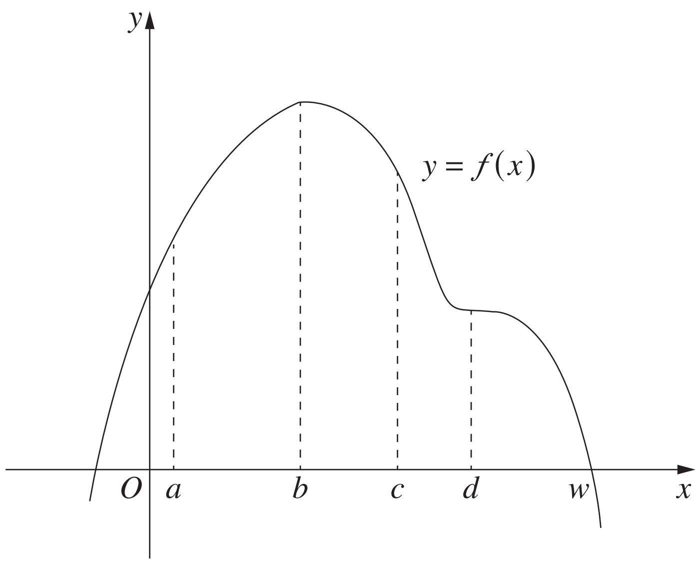
A. $P(t) = 1500 + 1500e^{-kt}$

B. $P(t) = 3000 - 1500e^{-kt}$

C. $P(t) = 3000 + 1500e^{-kt}$

D. $P(t) = 4500 - 1500e^{-kt}$

**6** The diagram shows the graph of $y = f(x)$. The equation $f(x) = 0$ has a solution at $x = w$.




Newton's method can be used to give an approximation close to the solution $x = w$.

Which initial approximation, $x_1$, will give the second approximation that is closest to the solution $x = w$?

A. $x_1 = a$

B. $x_1 = b$

C. $x_1 = c$

D. $x_1 = d$

**7** The velocity of a particle, in metres per second, is given by $v = x^2 + 2$, where $x$ is its displacement in metres from the origin.

What is the acceleration of the particle at $x = 1$?

A. $2\ \mathrm{m\,s^{-2}}$

B. $3\ \mathrm{m\,s^{-2}}$

C. $6\ \mathrm{m\,s^{-2}}$

D. $12\ \mathrm{m\,s^{-2}}$

**8** Six men and six women are to be seated at a round table.

In how many different ways can they be seated if men and women alternate?

A. $5!\ 5!$

B. $5!\ 6!$

C. $2!\ 5!\ 5!$

D. $2!\ 5!\ 6!$

**9** Which of the following is a general solution of the equation $\sin 2x = -\frac{1}{2}$?

A. $x = n\pi + (-1)^n \frac{\pi}{12}$

B. $x = \frac{n\pi}{2} + (-1)^n \frac{\pi}{12}$

C. $x = n\pi + (-1)^{n+1} \frac{\pi}{12}$

D. $x = \frac{n\pi}{2} + (-1)^{n+1} \frac{\pi}{12}$

**10** A particle is moving in simple harmonic motion. The displacement of the particle is $x$ and its velocity, $v$, is given by the equation $v^2 = n^2\left(2kx - x^2\right)$, where $n$ and $k$ are constants. The particle is initially at $x = k$.

Which function, in terms of time $t$, could represent the motion of the particle?

A. $x = k\cos(nt)$

B. $x = k\sin(nt) + k$

C. $x = 2k\cos(nt) - k$

D. $x = 2k\sin(nt) + k$

## Section II

**60 marks**
**Attempt Questions 11–14**
**Allow about 1 hour and 45 minutes for this section**

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

In Questions 11–14, your responses should include relevant mathematical reasoning and/or calculations.

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**Question 11** (15 marks) Use the Question 11 Writing Booklet.

(a) Consider the polynomial $P(x) = x^3 - 2x^2 - 5x + 6$.

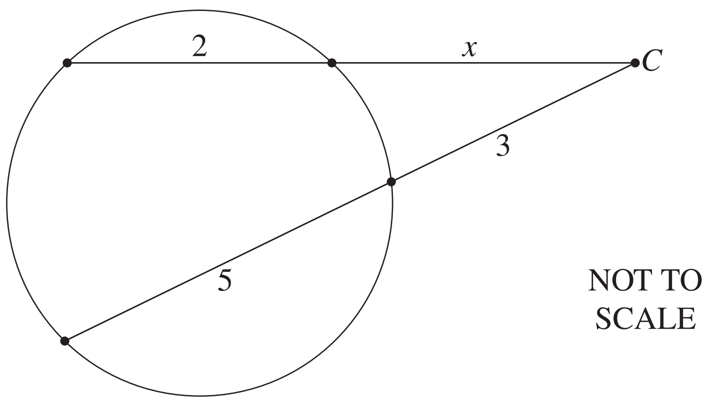
(i) Show that $x = 1$ is a zero of $P(x)$. **1**

(ii) Find the other zeros. **2**

(b) Solve $\log_2 5 + \log_2 (x - 2) = 3$. **2**

(c) Write $\sqrt{3}\sin x + \cos x$ in the form $R\sin(x + \alpha)$ where $R > 0$ and $0 \le \alpha \le \dfrac{\pi}{2}$. **2**

(d) Two secants from the point $C$ intersect a circle as shown in the diagram. **2**




What is the value of $x$?

**Question 11 continues on page 7**

Question 11 (continued)

(e) Consider the function $f(x) = \dfrac{1}{4x-1}$.

(i) Find the domain of $f(x)$. **1**

(ii) For what values of $x$ is $f(x) < 1$? **2**

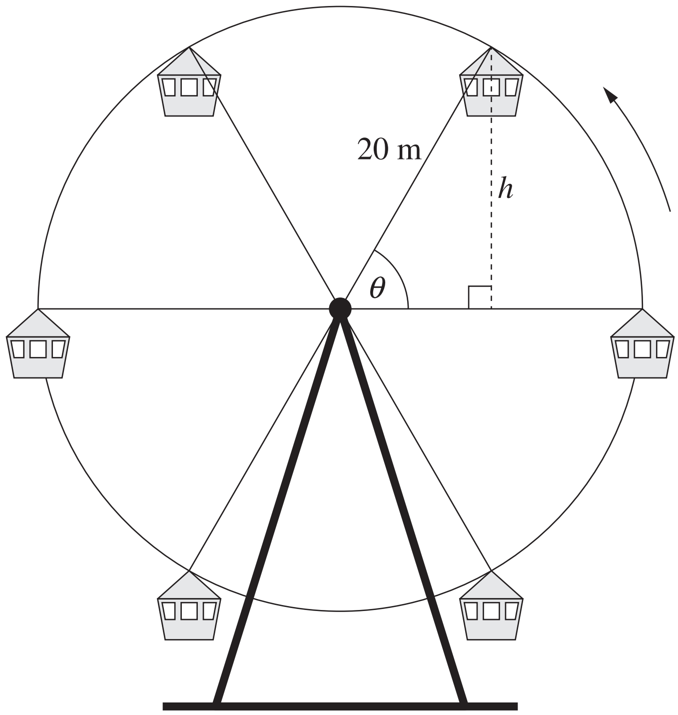
(f) Evaluate $\displaystyle\int_{-3}^{0} \frac{x}{\sqrt{1-x}}\,dx$, using the substitution $u = 1 - x$. **3**

**End of Question 11**

**Question 12** (15 marks) Use the Question 12 Writing Booklet.

(a) Find $\int \cos^2(3x)\,dx$. **2**

(b) A ferris wheel has a radius of 20 metres and is rotating at a rate of 1.5 radians per minute. The top of a carriage is $h$ metres above the horizontal diameter of the ferris wheel. The angle of elevation of the top of the carriage from the centre of the ferris wheel is $\theta$.




(i) Show that $\dfrac{dh}{d\theta} = 20\cos\theta$. **1**

(ii) At what speed is the top of the carriage rising when it is 15 metres higher than the horizontal diameter of the ferris wheel? Give your answer correct to one decimal place. **2**

(c) Let $f(x) = \sin^{-1}x + \cos^{-1}x$.

(i) Show that $f'(x) = 0$. **1**

(ii) Hence, or otherwise, prove $\sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2}$. **1**

(iii) Hence, sketch $f(x) = \sin^{-1}x + \cos^{-1}x$. **1**

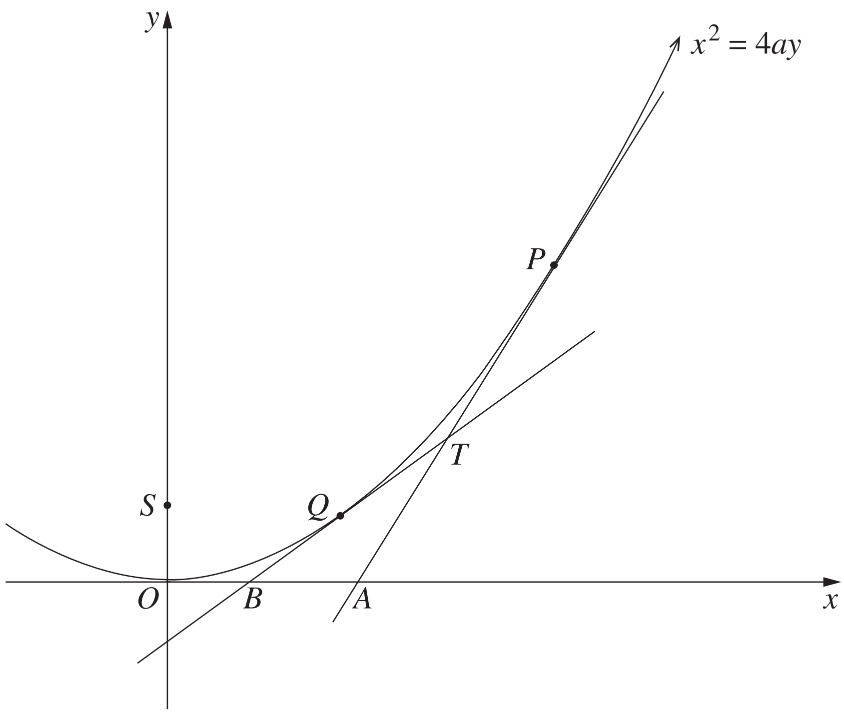
**Question 12 continues on page 9**

(d) A group of 12 people sets off on a trek. The probability that a person finishes the trek within 8 hours is 0.75. **2**

Find an expression for the probability that at least 10 people from the group complete the trek within 8 hours.

(e) The points $P(2ap,\ ap^2)$ and $Q(2aq,\ aq^2)$ lie on the parabola $x^2 = 4ay$. The focus of the parabola is $S(0,\ a)$ and the tangents at $P$ and $Q$ intersect at $T\left(a(p+q),\ apq\right)$. (Do NOT prove this.)

The tangents at $P$ and $Q$ meet the $x$-axis at $A$ and $B$ respectively, as shown.




(i) Show that $\angle PAS = 90°$. **2**

(ii) Explain why $S$, $B$, $A$, $T$ are concyclic points. **1**

(iii) Show that the diameter of the circle through $S$, $B$, $A$ and $T$ has length **2**

$$a\sqrt{\left(p^2+1\right)\left(q^2+1\right)}.$$

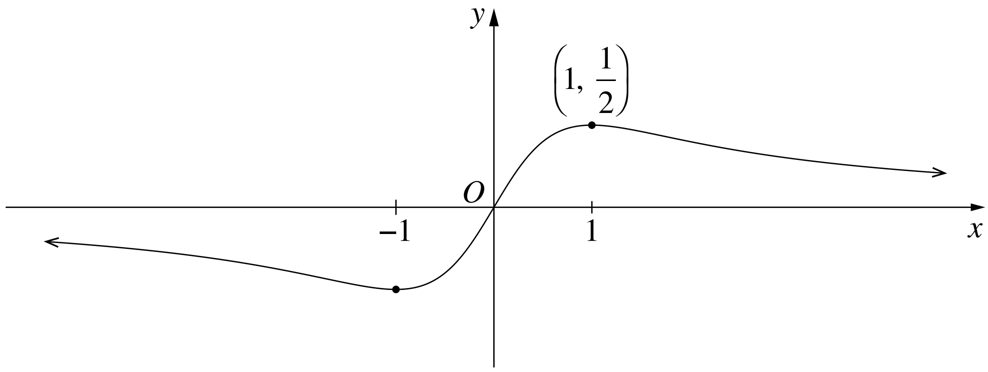
**End of Question 12**

**Question 13** (15 marks) Use the Question 13 Writing Booklet.

(a) Prove by mathematical induction that, for $n \geq 1$, **3**

$$2 - 6 + 18 - 54 + \cdots + 2(-3)^{n-1} = \frac{1-(-3)^n}{2}.$$

(b) The diagram shows the graph $y = \dfrac{x}{x^2+1}$, for all real $x$.




Consider the function $f(x) = \dfrac{x}{x^2+1}$, for $x \geq 1$.

The function $f(x)$ has an inverse. (Do NOT prove this.)

(i) State the domain and range of $f^{-1}(x)$. **2**

(ii) Sketch the graph $y = f^{-1}(x)$. **1**

(iii) Find an expression for $f^{-1}(x)$. **3**

**Question 13 continues on page 11**

(c) An object is projected from the origin with an initial velocity of $V$ at an angle $\theta$ to the horizontal. The equations of motion of the object are

$$x(t) = Vt\cos\theta$$
$$y(t) = Vt\sin\theta - \frac{gt^2}{2}. \quad \text{(Do NOT prove these.)}$$

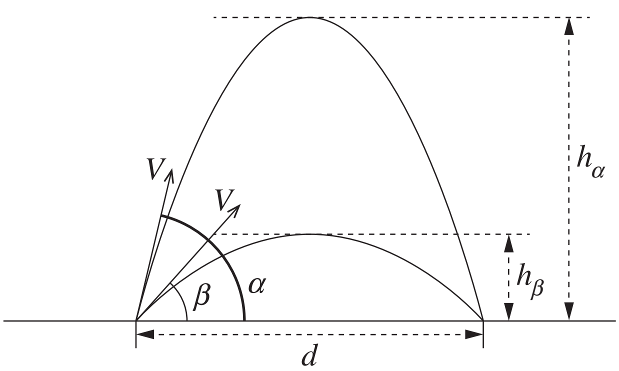
(i) Show that when the object is projected at an angle $\theta$, the horizontal range is $\dfrac{V^2}{g}\sin 2\theta$. **2**

(ii) Show that when the object is projected at an angle $\dfrac{\pi}{2} - \theta$, the horizontal range is also $\dfrac{V^2}{g}\sin 2\theta$. **1**

(iii) The object is projected with initial velocity $V$ to reach a horizontal distance $d$, which is less than the maximum possible horizontal range. There are two angles at which the object can be projected in order to travel that horizontal distance before landing. Let these angles be $\alpha$ and $\beta$, where $\beta = \dfrac{\pi}{2} - \alpha$. **3**

Let $h_\alpha$ be the maximum height reached by the object when projected at the angle $\alpha$ to the horizontal.

Let $h_\beta$ be the maximum height reached by the object when projected at the angle $\beta$ to the horizontal.

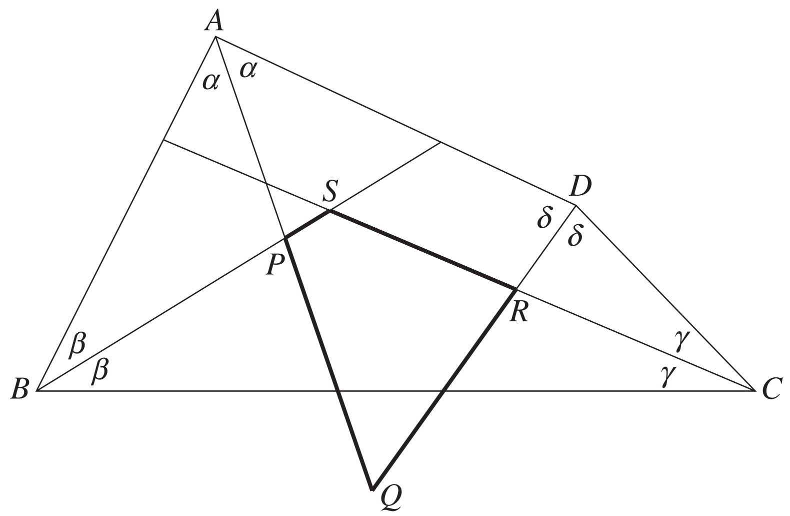



Show that the average of the two heights, $\dfrac{h_\alpha + h_\beta}{2}$, depends only on $V$ and $g$.

**End of Question 13**

**Question 14** (15 marks) Use the Question 14 Writing Booklet.

(a) The diagram shows quadrilateral $ABCD$ and the bisectors of the angles at $A$, $B$, $C$ and $D$. The bisectors at $A$ and $B$ intersect at the point $P$. The bisectors at $A$ and $D$ meet at $Q$. The bisectors at $C$ and $D$ meet at $R$. The bisectors at $B$ and $C$ meet at $S$. **3**




Copy or trace the diagram into your writing booklet.

Show that $PQRS$ is a cyclic quadrilateral.

(b) (i) By considering the expansions of $(1+(1+x))^n$ and $(2+x)^n$, show that **3**

$$\binom{n}{r}\binom{r}{r}+\binom{n}{r+1}\binom{r+1}{r}+\binom{n}{r+2}\binom{r+2}{r}+\cdots+\binom{n}{n}\binom{n}{r}=\binom{n}{r}2^{n-r}.$$

(ii) There are 23 people who have applied to be selected for a committee of 4 people. **2**

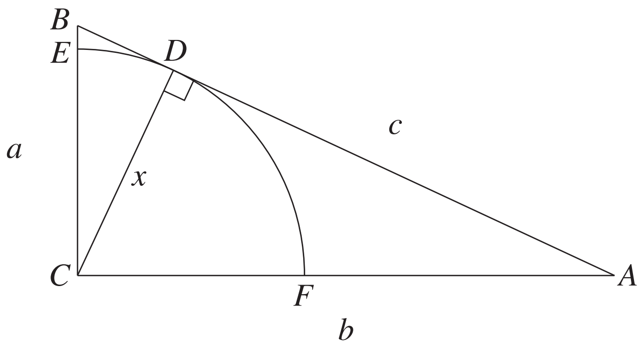
The selection process starts with Selector $A$ choosing a group of at least 4 people from the 23 people who applied.

Selector $B$ then chooses the 4 people to be on the committee from the group Selector $A$ has chosen.

In how many ways could this selection process be carried out?

**Question 14 continues on page 13**

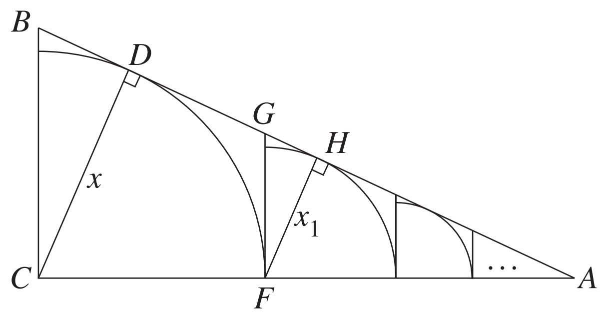
(c) In triangle $ABC$, $BC$ is perpendicular to $AC$. Side $BC$ has length $a$, side $AC$ has length $b$ and side $AB$ has length $c$. A quadrant of a circle of radius $x$, centred at $C$, is constructed. The arc meets side $BC$ at $E$. It touches the side $AB$ at $D$, and meets side $AC$ at $F$. The interval $CD$ is perpendicular to $AB$.




(i) Show that $\triangle ABC$ and $\triangle ACD$ are similar. **1**

(ii) Show that $x = \dfrac{ab}{c}$. **1**

From $F$, a line perpendicular to $AC$ is drawn to meet $AB$ at $G$, forming the right-angled triangle $GFA$. A new quadrant is constructed in triangle $GFA$ touching side $AB$ at $H$. The process is then repeated indefinitely.




(iii) Show that the limiting sum of the areas of all the quadrants is $\dfrac{\pi ab^2}{4(2c - a)}$. **4**

(iv) Hence, or otherwise, show that $\dfrac{\pi}{2} < \dfrac{2c - a}{b}$. **1**

**End of paper**

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# REFERENCE SHEET

– Mathematics –

– Mathematics Extension 1 –

– Mathematics Extension 2 –

# Mathematics

**Factorisation**

$$a^2 - b^2 = (a+b)(a-b)$$

$$a^3 + b^3 = (a+b)\left(a^2 - ab + b^2\right)$$

$$a^3 - b^3 = (a-b)\left(a^2 + ab + b^2\right)$$

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**Angle sum of a polygon**

$$S = (n-2) \times 180^\circ$$

---

**Equation of a circle**

$$(x-h)^2 + (y-k)^2 = r^2$$

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**Trigonometric ratios and identities**

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}}$$

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}$$

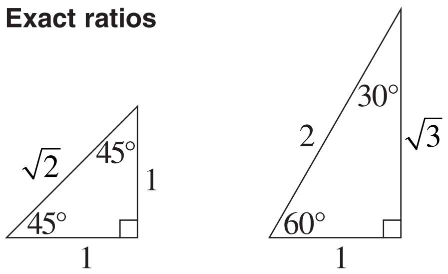
$$\sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\sin^2\theta + \cos^2\theta = 1$$

**Exact ratios**




**Sine rule**

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

**Cosine rule**

$$c^2 = a^2 + b^2 - 2ab\cos C$$

**Area of a triangle**

$$\text{Area} = \frac{1}{2}ab\sin C$$

**Distance between two points**

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

**Perpendicular distance of a point from a line**

$$d = \frac{\left|ax_1 + by_1 + c\right|}{\sqrt{a^2 + b^2}}$$

**Slope (gradient) of a line**

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

**Point–gradient form of the equation of a line**

$$y - y_1 = m(x - x_1)$$

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***n*th term of an arithmetic series**

$$T_n = a + (n-1)d$$

**Sum to *n* terms of an arithmetic series**

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right] \quad \text{or} \quad S_n = \frac{n}{2}(a + l)$$

***n*th term of a geometric series**

$$T_n = ar^{n-1}$$

**Sum to *n* terms of a geometric series**

$$S_n = \frac{a\left(r^n - 1\right)}{r - 1} \quad \text{or} \quad S_n = \frac{a\left(1 - r^n\right)}{1 - r}$$

**Limiting sum of a geometric series**

$$S = \frac{a}{1-r}$$

**Compound interest**

$$A_n = P\left(1 + \frac{r}{100}\right)^n$$

**Differentiation from first principles**

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

**Derivatives**

If $y = x^n$, then $\frac{dy}{dx} = nx^{n-1}$

If $y = uv$, then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

If $y = \frac{u}{v}$, then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

If $y = F(u)$, then $\frac{dy}{dx} = F'(u)\frac{du}{dx}$

If $y = e^{f(x)}$, then $\frac{dy}{dx} = f'(x)e^{f(x)}$

If $y = \log_e f(x) = \ln f(x)$, then $\frac{dy}{dx} = \frac{f'(x)}{f(x)}$

If $y = \sin f(x)$, then $\frac{dy}{dx} = f'(x)\cos f(x)$

If $y = \cos f(x)$, then $\frac{dy}{dx} = -f'(x)\sin f(x)$

If $y = \tan f(x)$, then $\frac{dy}{dx} = f'(x)\sec^2 f(x)$

---

**Solution of a quadratic equation**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

**Sum and product of roots of a quadratic equation**

$$\alpha + \beta = -\frac{b}{a} \qquad \alpha\beta = \frac{c}{a}$$

**Equation of a parabola**

$$(x-h)^2 = \pm 4a(y-k)$$

**Integrals**

$$\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$$

$$\int e^{ax+b} \, dx = \frac{1}{a}e^{ax+b} + C$$

$$\int \frac{f'(x)}{f(x)} dx = \ln\left| f(x) \right| + C$$

$$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + C$$

$$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C$$

$$\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + C$$

**Trapezoidal rule (one application)**

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2}\left[ f(a) + f(b) \right]$$

**Simpson's rule (one application)**

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]$$

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**Logarithms – change of base**

$$\log_a x = \frac{\log_b x}{\log_b a}$$

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**Angle measure**

$180° = \pi$ radians

**Length of an arc**

$l = r\theta$

**Area of a sector**

$$\text{Area} = \frac{1}{2}r^2\theta$$

# Mathematics Extension 1

**Angle sum identities**

$$\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

$$\tan(\theta+\phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$$

***t* formulae**

If $t = \tan\frac{\theta}{2}$, then

$$\sin\theta = \frac{2t}{1+t^2}$$

$$\cos\theta = \frac{1-t^2}{1+t^2}$$

$$\tan\theta = \frac{2t}{1-t^2}$$

**General solution of trigonometric equations**

$\sin\theta = a, \qquad \theta = n\pi + (-1)^n \sin^{-1} a$

$\cos\theta = a, \qquad \theta = 2n\pi \pm \cos^{-1} a$

$\tan\theta = a, \qquad \theta = n\pi + \tan^{-1} a$

---

**Division of an interval in a given ratio**

$$\left( \frac{mx_2 + nx_1}{m+n}, \ \frac{my_2 + ny_1}{m+n} \right)$$

---

**Parametric representation of a parabola**

For $x^2 = 4ay$,

$$x = 2at, \quad y = at^2$$

At $\left(2at, \ at^2\right)$,

tangent: $y = tx - at^2$

normal: $x + ty = at^3 + 2at$

At $\left(x_1, \ y_1\right)$,

tangent: $xx_1 = 2a\left(y + y_1\right)$

normal: $y - y_1 = -\frac{2a}{x_1}\left(x - x_1\right)$

Chord of contact from $\left(x_0, \ y_0\right)$: $xx_0 = 2a\left(y + y_0\right)$

**Acceleration**

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

**Simple harmonic motion**

$$x = b + a\cos\left(nt + \alpha\right)$$

$$\ddot{x} = -n^2\left(x - b\right)$$

---

**Further integrals**

$$\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{1}{a^2 + x^2}\, dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

---

**Sum and product of roots of a cubic equation**

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

$$\alpha\beta\gamma = -\frac{d}{a}$$

**Estimation of roots of a polynomial equation**

Newton's method

$$x_2 = x_1 - \frac{f\left(x_1\right)}{f'\left(x_1\right)}$$

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**Binomial theorem**

$$\left(a+b\right)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$